

# Quantum Phase Transitions in the $U(5) - O(6)$ Large $N$ limit

Feng Pan,<sup>1,2</sup> Yu Zhang,<sup>1</sup> and J. P. Draayer<sup>2</sup>

<sup>1</sup>Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China

<sup>2</sup>Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA

**Abstract** *The  $U(5) - O(6)$  transitional behavior of the Interacting Boson Model in the large  $N$  limit is revisited. Some low-lying energy levels, overlaps of the ground state wavefunctions,  $B(E2)$  transition rate for the decay of the first excited energy level to the ground state, and the order parameters are calculated for different total numbers of bosons. The results show that critical behaviors of these quantities are greatly enhanced with increasing of the total number of bosons  $N$ , especially fractional occupation probability for  $d$  bosons in the ground state, the difference between the expectation value of  $n_d$  in the first excited  $0^+$  state and the ground state, and another quantity related to the isomer shift behave similarly in both the  $O(6) - U(5)$  large  $N$  and  $U(5) - SU(3)$  phase transitions.*

**Keywords:** Phase transitions, order parameters, large  $N$  limit.

**PACS numbers:** 21.60.Fw, 05.70.Fh, 21.10.Re, 27.70.+q

Quantum phase transitions have been attracting a lot of attention in many areas of physics. This is understandable because they are very important for gaining a deeper understanding of various quantum many-body systems.<sup>[1–3]</sup> In atomic nuclei, such quantum phase transitions can be related to different geometrical shapes of the system, which can be described either by the Bohr-Mottelson model<sup>[4]</sup> (BMM) or by the Interacting Boson Model<sup>[3]</sup> (IBM). As summarized by Iachello,<sup>[5]</sup> the study of shape phase transitions in atomic nuclei was initiated in the early 80s<sup>[6–8]</sup> following some previous work<sup>[9]</sup> by Gilmore. It is now widely accepted that the three limiting cases of the IBM correspond to three different geometric shapes of nuclei, referred to as spherical (vibrational limit with  $U(5)$  symmetry), axially deformed (rotational limit with  $SU(3)$  symmetry), and  $\gamma$ -soft (triaxial with  $O(6)$  symmetry). This picture is captured by the so-called Casten triangle.<sup>[10]</sup> More interesting scenarios occur when a system is in between two different phases, in which case a quantum phase transition occurs at the corresponding critical point. A critical point at finite  $N$  with  $E(5)$  symmetry along the  $U(5) - O(6)$  leg of the Casten triangle was shown to exist in [11], and many examples

confirming the nature of this transition in realistic nuclear system have been reported.<sup>[12]</sup> Recently this transitional region has been studied for relatively large  $N$  values and the results show that the critical point region becomes progressively narrower as the boson number  $N$  increases.<sup>[13,14]</sup> This phenomenon has been explained in [14] in terms of a quasidynamical symmetry. In order to study the large  $N$  limit situation corresponding to the classical BMM, one must approach the large  $N$  limit from results for finite  $N$  if algebraic results for the large  $N$  limit is not available. In this Letter we revisit the  $U(5) - O(6)$  transitional case in the large  $N$  in detail to see whether there are substantial changes that occur as  $N$  grows ever larger, which serves as a supplement to the results reported in [13] and [14].

Our investigation is based on the following schematic  $U(5) - O(6)$  Hamiltonian:

$$H = (1 - x)\hat{n}_d + \frac{x}{f(N)}\hat{S}^+\hat{S}^-, \quad (1)$$

where  $\hat{n}_d = \sum_m d_m^\dagger d_m$  is the total number of  $d$ -bosons,  $\hat{S}^+ = \frac{1}{2}(d^\dagger \cdot d^\dagger - s^{\dagger 2})$  and  $\hat{S}^- = \frac{1}{2}(\tilde{d} \cdot \tilde{d} - s^2)$  are generalized boson pair creation and annihilation operators,  $f(N)$  is a linear function of total number of bosons  $N$ , and  $x$  is the control parameter of the model. It should be obvious that the system is in the  $U(5)$  limit when  $x = 0$  and in the  $O(6)$  limit when  $x = 1$ . As the control parameter  $x$  varies continuously within the closed interval  $[0, 1]$ , the system described by (1) undergoes a shape (phase) transition from  $U(5)$  to  $O(6)$ .

To diagonalize Hamiltonian (1), we expand the eigenstates of (1) in terms of the  $U(6) \supset U(5) \supset O(5) \supset O(3)$  basis vectors  $|N \ n_d \ v \ LM\rangle$  as

$$|N \ \xi \ v \ LM; \ x\rangle = \sum_{n_d} C_{n_d}^\xi(x) |N \ n_d \ v \ LM\rangle, \quad (2)$$

where  $C_{n_d}^\xi(x)$  is the expansion coefficient,  $\xi$  is an additional quantum number needed to label different eigenstates with the same quantum numbers  $v$ ,  $L$ , and,  $M$ .

To show how the energy levels change as a function of the control parameter  $x$  and the total number of bosons  $N$ , the lowest 25 energy levels as a function of  $x$  for a system with fixed quantum number  $\nu$  and  $f(N) = N$  for  $N = 10, 40, 120$ , and 300 are shown in Fig. 1. It can be seen from these results that there is a minimum in the low-lying excitation energy when the control parameter has a value in the range  $0.45 < x < 0.65$ , with the minimum growing sharper as the total number of bosons increases. This control parameter region is recognized as the critical point region of the vibrational-gamma soft transition. To the left of the critical point region,  $0 \leq x < 0.45$ , there are 9 degenerate levels ( $x = 0$ ) that gradually split with increasing  $x$  into 25 non-degenerate levels.

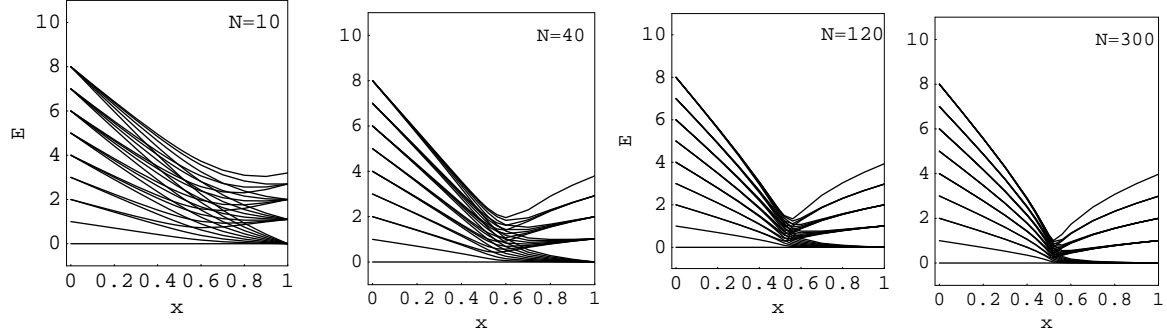


Fig 1. The lowest 25 energy levels (in arbitrary unit) of Hamiltonian (1) with  $f(N) = N$  as a function of  $x$  for  $N = 10, 40, 120$ , and  $300$ , respectively.

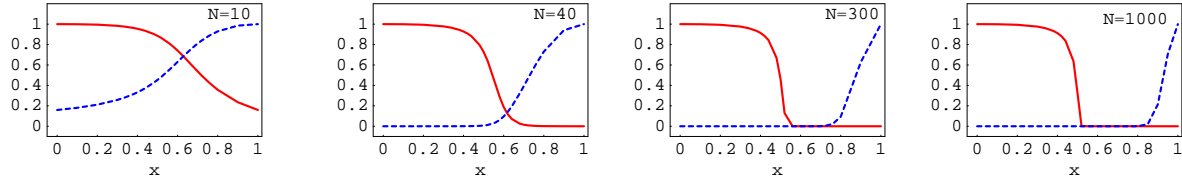


Fig. 2. Overlaps of the ground state wavefunction, where the full line shows the overlap  $|\langle 0_g; x|0_g; x=0 \rangle|$ , and the dotted line shows the overlap  $|\langle 0_g; x|x_g; x=1 \rangle|$ .

Similarly, beyond the critical region,  $0.65 < x \leq 1$ , the 25 non-degenerate levels coalesce into 5 degenerate levels ( $x = 1$ ). It should be noted that apart from the end points, the levels are truly non-degenerate, and that the level density grows rather dramatically within the critical point region with increasing  $N$ ; and furthermore, as  $N$  grows the critical point region becomes progressively narrower with a cusp around  $x \sim 0.45$ , which is in agreement with the observation reported in [14], in which only the  $N = 40$  case was shown.

The corresponding overlaps of the ground state wavefunctions of Hamiltonian (1) as a function of the control parameter  $x$  with those of limiting cases  $|\langle 0_g; x|0_g; x_0 \rangle|$  for  $x_0 = 0, 1$  for different total number of bosons  $N = 10, 40, 300, 1000$  were also calculated and the results are shown in Fig. 2. It can be seen from Fig. 2 that there is a cross-over point at a certain nonzero amplitude around  $x \sim 0.65$  for the overlaps  $|\langle 0_g; x|0_g; x=0 \rangle|$  and  $|\langle 0_g; x|x_g; x=1 \rangle|$  when  $N$  is relatively small, which yields to a cross-over region with near zero amplitude when  $N$  becomes larger. Furthermore, there is a sharp change in  $|\langle 0_g; x|0_g; x=0 \rangle|$  around a critical point  $x_c \sim 0.45$  in the large  $N$  limit. These results suggest that the largest absolute value of the derivative of  $|\langle 0_g; x|0_g; x=0 \rangle|$  with

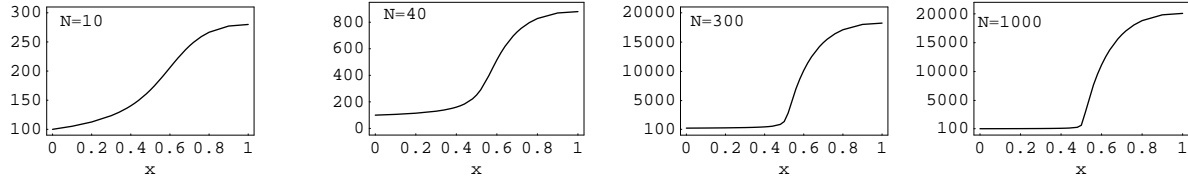


Fig. 3.  $B(E2)$  transition rates for decay of the first excited  $\nu = 1$  energy level to the ground state for  $N = 10, 40, 300$ , and  $1000$  expressed in units with  $B(E2; 1 \rightarrow 0) = 100$  in the  $U(5)$  ( $x = 0$ ) limit.

respect to  $x$  occurs around the critical point in the large  $N$  limit. While both  $|\langle 0_g; x | 0_g; x = 0 \rangle|$  and  $|\langle 0_g; x | 0_g; x = 1 \rangle|$  are all rather smooth in the relatively small  $N$  cases.

$B(E2)$  transition rates for decay of the first excited  $\nu = 1$  energy level to the ground state for  $N = 10, 40, 300$ , and  $1000$  expressed in units with  $B(E2; 1 \rightarrow 0) = 100$  in the  $U(5)$  ( $x = 0$ ) limit were also calculated. The  $E2$  transition operator was chosen as  $T(E2) = e_2(s^\dagger \tilde{d} + d^\dagger \tilde{s})_q^2$ , where  $e_2$  is the effective charge. The results are shown in Fig. 3. It can be seen quite clearly that the  $B(E2; 1 \rightarrow 0)$  changes rather smoothly with  $x$  for small  $N$ , while there is a sharp change at the critical point when  $N$  is large enough. This behavior of the  $B(E2; 1 \rightarrow 0)$  was also reported for the  $N \leq 60$  cases considered in [14].

The fractional occupation probability for  $d$  bosons in the ground state,  $\rho_d = \langle \hat{n}_d \rangle / N$  as a function of  $x$  was reported in [13] and [15]. It was shown that an order parameter to signify a second-order phase transition can be chosen to be  $\rho_d$ . Our calculation indicates that the system is almost in the  $U(5)$  limit when  $x \sim 0 - 0.45$  in the large  $N$  limit, which corresponds to an  $s$ -boson condensate. The occupation probability  $\rho_d$  gradually increases within the critical region for relatively small  $N$  values with the change in  $\rho_d$  becoming sharper and sharper with increasing  $N$ , which is in agreement to the results reported in [13] and [15]. Since the behavior of the order parameter  $\rho_d$  is the same for both first- and second-order transitions for the small  $N$  cases, in order to distinguish whether the phase transition is of first or second order from model calculations, another order parameter, the difference between the expectation value of  $n_d$  in the first excited  $0^+$  state and the ground state,  $v_1 = \alpha_0(\langle 0_2 | n_d | 0_2 \rangle - \langle 0_g | n_d | 0_g \rangle)$ , was introduced in [15]. The authors showed that  $v_1$  displays a wiggling, sign-change-behavior in the region of the critical point due to the switching of the two coexisting phases, which is characteristic of a first-order transition, while  $v_1$  has a smoother behavior that is characteristic of a second-order transition. It should be pointed out that the conclusion made in [15] are for finite  $N$  only. However, since order of a phase transition should always be defined in the thermodynamic limit, an effective order parameter must also behave

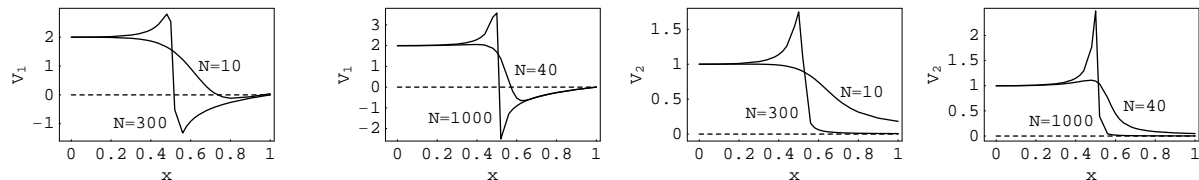


Fig. 4. Behavior of the order parameters  $v_1$  and  $v_2$  as functions of the control parameter  $x$  for different  $N$  values, where the parameters  $\alpha_0$  and  $\beta_0$  in  $v_1$  and  $v_2$ , respectively, are set to be 1.

differently in phase transitions with different orders. To see whether the order parameter  $v_1$  and another quantity  $v_2 = \beta_0(\langle 2_1 | n_d | 2_1 \rangle - \langle 0_1 | n_d | 0_1 \rangle)$  related to the isomer shift  $\delta \langle r^2 \rangle = \langle r^2 \rangle_{2_1} - \langle r^2 \rangle_{0_g}$  introduced in [15] satisfy this criterion, both  $v_1$  and  $v_2$  were calculated for the  $N = 10, 40, 300$ , and 1000 cases. The results are shown in Fig. 4. In order to compare curves of  $v_1$  and  $v_2$  for the different  $N$  cases, the parameters  $\alpha_0$  and  $\beta_0$  were taken to be 1 instead of the  $1/N$  used in [15]. Our calculation shows that: (a) both  $v_1$  and  $v_2$  have a smooth behavior when  $N$  is relatively small; (b)  $v_1$  gradually displays of a sign-changing nature in the critical region when  $N$  is large enough, with this behavior being greatly enhanced in the large  $N$  limit; and (c) there is an obvious peak in  $v_2$  in the large  $N$  limit, while  $v_2$  is rather smooth for relatively small  $N$ . These results shown in Fig. 4, together with those shown in [15], indicate that the order parameters  $v_1$  and  $v_2$ , like another order parameter  $\rho_d$ , behave similarly in both the  $O(6) - U(5)$  large  $N$  and  $U(5) - SU(3)$  phase transitions. Due to current computation limitation, one can not calculate these quantities in the  $U(5) - SU(3)$  transitional case exactly for  $N \geq 30$ . Therefore, whether these quantities in the  $U(5) - SU(3)$  case will change substantially in the large  $N$  limit is still an open question. For relatively small  $N$  cases, however, as indicated in [15], the order parameters  $v_1$  and  $v_2$  are indeed behave differently in the  $O(6) - U(5)$  and  $U(5) - SU(3)$  phase transitions, which, therefore, can be used to signify the order of the transition from the small  $N$  cases.

In summary, the  $O(6) - U(5)$  transitional behavior in the large  $N$  limit has been revisited. Some low-lying energy levels, overlaps of the ground state wavefunctions,  $B(E2)$  transition rates for decay of the first excited  $\nu = 1$  energy level to the ground state, and the order parameters  $v_1$  and  $v_2$  related to the isomer shifts were calculated for different total number of bosons. It is found that the critical behaviors of these quantities are greatly enhanced with increasing of the total number of bosons  $N$ , especially all the order parameters,  $\rho_d$ ,  $v_1$ , and  $v_2$  behave similarly in both the  $O(6) - U(5)$  large  $N$  and  $U(5) - SU(3)$  phase transitions. The drastic enhancement of these quantities near the critical point may be explained in terms of a quasidynamical symmetry.<sup>[14]</sup> The

“specific heat” introduced in [16] seems also suitable to be used to classify the order of the phase transitions since these quantities behave quite differently in first and second order phase transitions even when the  $N$  is finite.

In the IBM for atomic nuclei the total number of bosons  $N$  is phenomenologically related to be the number of valence  $s$  and  $d$  nucleon pairs, which is usually a relatively small number. However, in the large  $N$  limit, the IBM yields to the BMM, in which there is no restriction on the number of bosons; indeed, in principle, this should correspond to the  $N \rightarrow \infty$  limit. Therefore, the results shown in this Letter should be helpful in understanding the nature of the vibration to gamma-soft phase transition in the BMM. It is interesting to check to see whether there are substantial differences between the  $E(5)$  symmetry derived from an extreme case of the BMM and systems described by a  $U(5)-O(6)$  Hamiltonian with the finite  $N$  based on the IBM. A recent study suggest that the  $E(5)$  symmetry can only be described approximately in the IBM,<sup>[17]</sup> which is a conclusion that is consistent with our results.

Support from the U.S. National Science Foundation (0140300), the Southeastern Universities Research Association, the Natural Science Foundation of China (10175031), the Education Department of Liaoning Province, and the LSU-LNNU joint research program (C164063)) is acknowledged.

## References

- [1] S. Sachdev, *Quantum Phase Transitions* (Cambridge Univ. Press, Cambridge, 1999).
- [2] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
- [3] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
- [4] A. Borh and B. R. Mottelson, *Nuclear Structure* Vol. I (Benjamin, New York, 1969); Vol. II (Benjamin, New York, 1975).
- [5] F. Iachello, AIP Conf. Proc. **726**, 111 (2004).
- [6] A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980).
- [7] D. H. Feng, R. Gilmore, and S. R. Deans, Phys. Rev. **C23**, 1254 (1981).
- [8] O. S. Van Roosmalen, *Algebraic Description of Nuclear and Molecular Rotation-Vibration Spectra*, Ph.D. Thesis, University of Groningen, The Netherlands, 1982.
- [9] R. Gilmore and D. H. Feng, Nucl. Phys. **A301**, 189 (1978); R. Gilmore, J. Math. Phys. **20**, 89 (1979).
- [10] R. F. Casten, in *Interacting Bose-Fermi System*, ed. F. Iachello (Plenum, 1981).
- [11] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
- [12] R. M. Clark, M. Cromaz, M. A. Deleplanque, M. Descovich, R. M. Diamond, P. Fallon, I. Y. Lee, A. O. Macchiavelli, H. Mahmud, E. Rodriguez-Vieitez, F. S. Stephens, and D. Ward, Phys. Rev. **C69**, 064322 (2004).

- [13] J. M. Arias, J. Dukelsky, and J. E. Garcia-Ramos, Phys. Rev. Lett. **91**, 162502 (2003).
- [14] D. J. Rowe, Phys. Rev. Lett. **93**, 122502 (2004).
- [15] F. Iachello, N. V. Zamfir, Phys. Rev. Lett. **92**, 212501 (2004).
- [16] P. Cejnar, S. Heinze, and J. Dobeš, Phys. Rev. C **71**, 011304 (2005).
- [17] J. M. Arias, C. E. Alonso, A. Vitturi, J. E. Garcia-Ramos, J. Dukelsky, A. Frank, Phys. Rev. **C68**, 041302 (2003).